Method of Maximum Likelihood Estimation

Definition:

This method states that to consider every possible value that the parameter might have and for each value compute the probability that the given sample would have occurred. If that were the true value of parameter then value of the parameter for which the probability of a given sample is greatest and is chosen as an estimate is called Maximum Likelihood Estimate and the procedure is called Maximum Likelihood Estimation.

Procedure:

First of all take the likelihood function of the p.d.f then take log of the likelihood function. Then partially differentiate log likelihood function with respect to parameter and put equal to zero. Then obtain the estimate of the respective parameter

i.e.

$$\frac{\partial \log L(\underline{x})}{\partial \theta} = 0$$

Then obtain $\hat{\theta}$

Note:

Every distribution by likelihood is partially differentiated with respect to its parameter separately. Then we get the respective parameter estimate. if $\hat{\theta}$ is the MLE of the parameter θ then its variance is given by

$$V(\hat{\theta}) = \frac{1}{-E\left[\partial^2 \log L(\underline{x};\theta)\right] / \partial\theta^2}$$

Information Matrix= $I_{i_j} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$

$$v_{11} = \frac{1}{-E\left[\partial^2 \log L(\underline{x};\mu)\right] / \partial\mu^2}$$

$$v_{22} = \frac{1}{-E\left[\partial^2 \log L(\underline{x};\delta^2)\right] \big/ \partial(\delta^2)^2}$$

$$v_{21}=v_{12} = \frac{1}{-E\left[\partial^2 \log L(\underline{x};\mu,\delta^2)\right] \big/ \partial\mu.\partial\delta^2}$$

Variance and covariance matrix$=(I_{ij})^{-1}$

$$= adj(I_{ij}) \big/ \det(I_{ij})$$

Leading diagonal values represent variances and other represent covariance of the estimates=

$$= \begin{bmatrix} \delta^2/n & 0 \\ 0 & 2\,\delta^2/n \end{bmatrix}$$

Properties of MLE:

1. Maximum Likelihood Estimators are consistent.
2. Maximum Likelihood Estimators are approximately normal such as n $\to\infty$ MLE becomes normal.
3. ML estimators are most efficient and variance of MLE becomes zero as n $\to\infty$.
4. If a sufficient estimator exists it can only be form by method of Maximum Likelihood function.
5. MLE are the function of sufficient statistic.
6. MLE are invariant under functional transformation.
7. MLE can only be obtained when density function is given.
8. ML estimators are not only unbiased.
9. Maximum Likelihood Estimators are consistent.
10. Maximum Likelihood Estimators are approximately normal such as n $\to\infty$ MLE becomes normal.
11. ML estimators are most efficient and variance of MLE becomes zero as n $\to\infty$.
12. If a sufficient estimator exists it can only be form by method of Maximum Likelihood function.
13. MLE are the function of sufficient statistic.
14. MLE are invariant under functional transformation.
15. MLE can only be obtained when density function is given.
16. ML estimators are not only unbiased.

Question#1

Suppose X=4 are fixed then by binomial formula the probability of x=4 when n=10.given that

$F(4,10,P) = C_4{}^{10} P^4 q^6$

Obtain the MLE for P

When

P=0.00,0.108,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0

Solution

No	P	$F(4,10,P) = C_4{}^{10} P^4 q^6$
0	0.00	0
1	0.108	0.0144
2	0.2	0.0880
3	0.3	0.20012
4	0.4	0.2508
5	0.5	0.2050
6	0.6	0.11147
7	0.7	0.03675
8	0.8	0.005505
9	0.9	0.00013778
10	1.0	0

We see that at P=0.4 has maximum probability therefore $\hat{P}$ =0.4 is MLE for P.

Question 2:

Let $X_1, X_2, X_3, ..., X_n$ be a random sample from the density f(x; θ)= ($(\theta+1)X^{\theta}$. Find the MLE for 'θ'. Also find variance.

As given that

f(x; θ) = $(\theta+1)X^{\theta}$

Taking likelihood function

L(x; θ) $=(\theta+1)^n \prod_{i=1}^{n} X^{\theta}$

Taking log function

$$\log L(X;\theta) = n\log(\theta+1) + \theta\sum\log X$$

Differentiate with respect to "θ"

$$\frac{\partial \log L(X;\theta)}{\partial\theta} = \frac{n}{(\theta+1)} + \sum\log X \to (A)$$

$$0 = n(\theta+1)^{-1} + \sum\log X$$

$$-\sum\log X = \frac{n}{\theta+1}$$

$$-\sum\log X \Big/ n = \frac{1}{\theta+1}$$

$$\theta+1 = -\frac{n}{\sum\log X}$$

$$\theta = -1 - \frac{n}{\sum\log X}$$

$$\theta = -1[\frac{\sum\log X + n}{\sum\log X}]$$

Again differentiate with respect to "θ"to eq (A)

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} = -\ \mathrm{n}(\theta+1)^{-2} + 0$$

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} -\ \mathrm{n}(\theta+1)^{-2}$$

Taking expectation on both sides

$$E[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}] = -\frac{n}{(\theta+1)^{-2}}$$

Now,

$$\operatorname{var}(\hat{\theta}) = \frac{1}{-E[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}]}$$

$$\operatorname{var}(\hat{\theta}) = \frac{1}{-(-\frac{n}{(\theta+1)^{-2}})}$$

$$\operatorname{var}(\hat{\theta}) = \frac{(\theta+1)^2}{n}$$

Required result

Question no 3:

Let $X_1, X_2, X_3,...., X_n$ **be a random sample from the density f(x;** θ**) =** $\frac{1}{\theta}e^{-\frac{x}{\theta}}$**. Find the MLE for '**$\theta$**'. Also find variance**.

As

f(x; θ) = $\frac{1}{\theta}e^{-\frac{x}{\theta}}$

Taking likelihood function

L(x; θ) $= (\frac{1}{\theta})^n e^{-\frac{\sum x}{\theta}}$

Taking log function

$\log L(X;\theta)$ = nlog $(\frac{1}{\theta}) - \frac{\sum x}{\theta}\log e$

$\log L(X;\theta)$ = -nlog $\theta - \frac{\sum x}{\theta}$

Differentiate with respect to "θ"

$$\frac{\partial \log L(X;\theta)}{\partial\theta} = -(\frac{n}{\theta}) + \frac{\sum x}{\theta^2} \rightarrow (A)$$

$$0 = -(\frac{n}{\theta}) + \frac{\sum x}{\theta^2}$$

$$\frac{-n\theta + \sum x}{\theta^2} = 0$$

$$-n\theta + \sum x = 0$$

$$\sum x = n\theta$$

$$\frac{\sum x}{n} = \theta$$

$$\hat{\theta} = \overline{x}$$

Again differentiate with respect to "θ" to eq (A)

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} = (\frac{n}{\theta^2}) - \frac{2\sum x}{\theta^3}$$

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} = (\frac{n}{\theta^2}) - \frac{2n\theta}{\theta^3} \quad \begin{array}{l} \therefore \overline{x} = \frac{\sum x}{n} \\ \therefore n\overline{x} = \sum x \\ \therefore n\theta = \sum x \end{array}$$

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} = (\frac{n}{\theta^2}) - \frac{2n}{\theta^3}$$

$$\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2} = -(\frac{n}{\theta^2})$$

Taking expectation on both sides

$$E[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}] = -(\frac{n}{\theta^2})$$

Now,

$$\mathrm{var}(\hat{\theta}) = \frac{1}{-E[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}]}$$

$$\mathrm{var}(\hat{\theta}) = \frac{1}{-(-\frac{n}{\theta^2})}$$

$$\mathrm{var}(\hat{\theta}) = \frac{\theta^2}{n}$$

This is the required result.

Question no 4:

Find an estimate of for a given density $f(x;\alpha) = \frac{2}{\alpha^2}(\alpha - x)\, l_{(0,\alpha)(x)}$ **for a sample of size '2'by using method of moments.**

Given the following p.d.f

$$f(x;\alpha) = \frac{2}{\alpha^2}(\alpha - x)\ 0 < x < \alpha$$

Generally resolve it with the help of order statistic because the parameter involve in the interval of p.d.f.

$$0 < X_{(1)} < X_{(2)} < \alpha$$

SO

$$\alpha = X_{(2)}$$

Method of moments:

By definition of sample moments

$$M'_r = E(X)^r$$

$$M'_r = \frac{\Sigma X}{n}$$

By definition of population moments

$$\mu'_r = E(X)^r$$

$$\mu'_1 = E(X)$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\alpha} x \frac{2}{\alpha^2} (\alpha - x) dx$$

$$= \frac{2}{\alpha^2} \int_0^{\alpha} x\alpha - x^2 dx$$

$$= \frac{2}{\alpha^2} \int_0^{\alpha} x\alpha - x^2 dx$$

$$= \frac{2}{\alpha^2} [\frac{\alpha^3}{2} + \frac{\alpha^3}{3}]$$

$$= \frac{2\alpha^3}{\alpha^2} [\frac{1}{2} + \frac{1}{3}]$$

$$= 2\alpha[\frac{3-2}{6}]$$

$$= \frac{2\alpha}{6}$$

$$= \frac{\alpha}{3}$$

Compare M_1' **and** μ_1'

$$\sum X / n = \frac{\alpha}{3}$$

$$3\sum X / n = \hat{\alpha}$$

$$3\overline{X} = \hat{\alpha}$$

Question no 5:

A random sample of size ‘n’ is drawn from a uniform or triangular or rectangular distribution having density function $f(x, \beta) = \frac{1}{\beta} \; 0 \le x \le \beta$

=0 otherwise

Find MLE for β .

Given p.d.f is

$$f(x,\beta)=\frac{1}{\beta}$$

Since parameter β is involve in the upper limit of density function therefore we cannot find MLE of β by using real procedure so we can use order statistic to find MLE of β .In this regard

i.e.

$$0<X_1<X_2<X_3<...<X_n<\beta$$

So MLE for β

$$\hat{\beta}=X_{(n)}$$

Question 6:

Let $X_1, X_{2,} X_3....X_n$ be a random sample from the density of –ve exponential $f(X;\theta)=\theta e^{-X\theta}$. Find MLE for θ and also find its variance.

Question 7:

If $f(X;\theta)=\theta e^{-X\theta}$ X>0 then show that $\hat{\theta}=\frac{1}{\overline{\overline{X}}}$ and $\text{var}=\frac{\theta^2}{n}$

Solution

As

$$f(X;\theta)=\theta e^{-X\theta}$$

Taking likelihood function

$$L(X;\theta)=\theta^n e^{-\theta\sum X}$$

Taking log function

$$\log L(X;\theta)=n\log\theta-\theta\sum X$$

Differentiate with respect to “θ”

$$\frac{\partial \log L(X;\theta)}{\partial \theta} = \frac{n}{\theta} - \sum X \;\rightarrow (A)$$

$$0 = \frac{n}{\theta} - \sum X$$

$$\sum X = \frac{n}{\theta}$$

$$\frac{n}{\theta} = \frac{1}{\sum X}$$

$$\theta = \frac{n}{\sum X}$$

$$\theta = \frac{1}{\frac{\sum X}{n}}$$

$$\hat{\theta} = \frac{1}{\overline{X}}$$

Again differentiate with respect to “θ”to eq (A)

$$\frac{\partial^2 \log L(X;\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} + 0$$

Taking expectation on both sides

$$E[\frac{\partial^2 \log L(X;\theta)}{\partial \theta^2}] = -\frac{n}{\theta^2}$$

NOW,

$$\operatorname{var}(\hat{\theta}) = \frac{1}{-E[\frac{\partial^2 \log L(X;\theta)}{\partial \theta^2}]}$$

$$= \frac{1}{-(-\frac{n}{\theta^2})}$$

$$\text{var}(\hat{\theta}) = \frac{\theta^2}{n}$$

Question 8:

A random sample of size n is taken from a normal population with mean and variance (μ, σ^2) , σ^2 is unknown obtain the MLE of" µ". Also find variance of estimator.

Solution

Given that

$$X \sim N(X; \mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\Pi}} e^{-\frac{1}{2}(\frac{X-\mu}{\sigma})^2} \quad -\infty \leq X \leq \infty$$

Taking likelihood function

$$L(X) = (\frac{1}{\sigma\sqrt{2\Pi}})^n e^{-\frac{1}{2\sigma^2}\sum(X-\mu)^2}$$

$$= (\frac{1}{\sqrt{2\Pi}})^n (\frac{1}{\sigma^2})^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(X-\mu)^2}$$

Taking log function

$$\log L(X) = n \log \frac{1}{\sqrt{2\Pi}} - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (X - \mu)^2$$

Differentiate with respect to "µ"

$$\frac{\partial \log L(X)}{\partial \mu} = 0 - 0 - \frac{1}{2\sigma^2} 2 \sum (X - \mu)(-1)$$

$$0 = \frac{\sum X - n\mu}{\sigma^2}$$

$$\sum X - n\mu = 0$$

$$\sum X = n\mu$$

$$\mu = \frac{\sum X}{n}$$

$$\hat{\mu} = \overline{X}$$

Again differentiate with respect to "µ"

$$\frac{\partial^2 \log L(X)}{\partial \mu^2} = \frac{-n}{\sigma^2}$$

Taking expectation on both sides

$$E[\frac{\partial^2 \log L(X)}{\partial \mu^2}] = \frac{-n}{\sigma^2}$$

Now,

$$\text{var}(\hat{\mu}) = \frac{1}{-E[\frac{\partial^2 \log L(X)}{\partial \mu^2}]}$$

$$= \frac{1}{-(-n/\sigma^2)}$$

$$\text{var}(\hat{\mu}) = \frac{\sigma^2}{n}$$

This is required.

Question 9:-

A random sample of size 'n' is taken from a normal population (μ, δ^2). Obtain the MLE of $'\mu'$ and $'\delta^2'$ find variance & covariance.

Given that

$$X \sim N(X\,; \mu, \delta^2)$$

$$f(x) = \frac{1}{\delta\sqrt{2\pi}}\; e^{\frac{1}{2\delta^2}(x-\mu)^2} \qquad\qquad -\infty \le x \le +\infty$$

Taking likelihood function

$$L(\underline{x}) = \left(\frac{1}{\delta\sqrt{2\pi}}\right)^n \; e^{-\frac{1}{2\delta^2}\Sigma(x-\mu)^2}$$

$$L(\underline{x}) = \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} \; \left(\frac{1}{\sqrt{2\pi}}\right)^n \; e^{-\frac{1}{2\delta^2}\Sigma(x-\mu)^2}$$

Take log likelihood function

Log L ($\underline{x}$) = n log $\left(\frac{1}{\sqrt{2\Pi}}\right)$ -n/2 log δ^2 - $\frac{1}{2\delta^2}\Sigma(x-\mu)^2$ $\qquad \to A$

Differentiate with respect to "μ"

$$\frac{\partial \log L(X)}{\partial \mu} = 0 - 0 - \frac{1}{2\sigma^2} 2\sum(X-\mu)(-1)$$

$$0 = \frac{\sum X - n\mu}{\sigma^2}$$

$$\sum X - n\mu = 0$$

$$\sum X = n\mu$$

$$\mu = \frac{\sum X}{n}$$

$$\hat{\mu} = \overline{X}$$

Again differentiate with respect to "μ"

$$\frac{\partial^2 \log L(X)}{\partial \mu^2} = \frac{-n}{\sigma^2}$$

Taking expectation on both sides

$$E[\frac{\partial^2 \log L(X)}{\partial\mu^2}] = \frac{-n}{\sigma^2}$$

Now,

$$\mathrm{var}(\hat{\mu}) = \frac{1}{-E[\frac{\partial^2 \log L(X)}{\partial\mu^2}]}$$

$$= \frac{1}{-(-n/\sigma^2)}$$

$$\mathrm{var}(\hat{\mu}) = \frac{\sigma^2}{n}$$

Now again differentiate with respect to δ^2 to equation (A) and put equal to zero.

$$\frac{\partial \log L(\underline{x})}{\partial\delta^2} = 0 - \frac{n}{2\delta^2} + \frac{1}{2\delta^4}\sum(x-\mu)^2$$

$$= \frac{\sum(x-\mu)^2}{2\delta^4} - \frac{n}{2\delta^2} \qquad \to (C)$$

$$\frac{2\delta^4 n}{2\delta^2} = \sum(x-\mu)^2$$

$$\hat{\delta}^2 = \frac{\sum(x-\mu)^2}{n}$$

Now again differentiate with respect to δ^2 to equation (C)

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{\partial^2}{\partial(\delta^2)^2}\left[\frac{\sum(x-\mu)^2}{2\delta^4} - \frac{n}{2\delta^2}\right]$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{\partial^2}{\partial(\delta^2)^2}\left[\frac{\sum(x-\mu)^2}{2(\delta^2)^2} - \frac{n}{2\delta^2}\right]$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{\partial^2}{\partial(\delta^2)^2}\left[\frac{\sum(x-\mu)^2}{2}(\delta^2)^{-2} - \frac{n}{2}(\delta^2)^{-1}\right]$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{-2\sum(x-\mu)^2}{2}(\delta^2)^{-3} + \frac{n}{2}(\delta^2)^{-2}$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{n}{2(\delta^4)} - \frac{\sum(x-\mu)^2}{(\delta^6)}$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{n}{2\delta^4} - \frac{n\delta^2}{\delta^6}$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2} = \frac{n}{2\delta^4} - \frac{n}{\delta^4} = \frac{n-2n}{2\delta^4} = \frac{-n}{2\delta^4}$$

Apply expectation on both sides.

$$E(\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2}) = \frac{-n}{2\delta^4}$$

$$V(\hat{\delta}^2) = \frac{1}{-E(\frac{\partial^2 \log L(\underline{x})}{\partial(\delta^2)^2})}$$

$$V(\hat{\delta}^2) = \frac{1}{-(\frac{-n}{2\delta^4})} = \frac{2\delta^4}{n}$$

For Covariance:-

$$\frac{\partial^2 \log L(\underline{x})}{\partial\mu\ \partial\delta^2} = -\left[\frac{\sum(x-\mu)}{\delta^4}\right] = -\left[\frac{\sum x - n\mu}{\delta^4}\right]$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial\mu\ \partial\delta^2} = \left[\frac{n\mu - \sum x}{\delta^4}\right] = \left[\frac{n\mu - n\overline{X}}{\delta^4}\right]$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial\mu\ \partial\delta^2} = 0$$

Question 10:

Obtain the MLE of the unknown parameter from the density

$$f(x) = (\frac{1}{\overline{)P}\ \theta^P})\, X^{p-1}\ e^{-\frac{x}{\theta}}$$

Where P is unknown and also obtains the variance of the estimator.

Let $x \sim G(P, \theta)$

$$f(x) = \left(\frac{1}{\Gamma P\, \theta^{P}}\right) X^{p-1}\; e^{-\frac{x}{\theta}}$$

Taking Likelihood Function

$$L(\underline{x}) = \left(\frac{1}{\Gamma P\, \theta^{P}}\right)^{n} \prod_{i=1}^{n} X^{P-1}\; e^{-\frac{\sum x}{\theta}}$$

Taking log likelihood function

$$\log L(\underline{x}) = n \log \frac{1}{\Gamma P} - n \log \theta^{P} + (P-1) \sum \log x - \frac{\sum x}{\theta}$$

$$\log L(\underline{x}) = n \log \frac{1}{\Gamma P} - nP \log \theta + (P-1) \sum \log x - \frac{\sum x}{\theta}$$

Differentiate with respect to 'θ' and put equal to zero.

$$\frac{\partial \log L(\underline{x})}{\partial \theta} = -\frac{nP}{\theta} + \frac{\sum x}{\theta^{2}} \quad \rightarrow A$$

$$0 = \frac{\sum x}{\theta^{2}} - \frac{nP}{\theta}$$

$$\frac{\sum x}{\theta^{2}} = \frac{nP}{\theta}$$

$$\frac{\sum x}{\theta^{2}}\, \theta = nP$$

$$\frac{\sum x}{\theta} = nP$$

$$\theta = \frac{\sum x}{nP}$$

$$\hat{\theta} = \frac{\overline{x}}{p}$$

Now again differentiate with respect to 'θ' equation (A)

$$\frac{\partial^2 \log L(\underline{x})}{\partial\theta^2} = \frac{nP}{\theta^2} - \frac{2\sum x}{\theta^3}$$

$$\frac{\partial^2 \log L(\underline{x})}{\partial\theta^2} = \frac{nP}{\theta^2} - \frac{2np\theta}{\theta^3} \qquad \therefore \frac{\sum x}{nP} = \theta \ , \ \sum x = nP\theta$$

$$= \frac{nP}{\theta^2} - \frac{2np}{\theta^2}$$

$$= -\frac{nP}{\theta^2}$$

Now,

$$V(\hat{\theta}) = \frac{1}{-E(\frac{\partial^2 \log L(\underline{x})}{\partial(\theta^2)})}$$

$$V(\hat{\theta}) = \frac{1}{-(-\frac{np}{\theta^2})}$$

$$V(\hat{\theta}) = \frac{\theta^2}{np}$$

Note:

If the parameter for which are find the MLE involve in the limits we cannot find the MLE by using the real procedure. If parameter involve in the lower limit then find first order statistic is its MLE. If parameter involve in the upper limit then nth order statistic will be its MLE.

Question 11:-

Given the density of raliegh distribution $f(x) = \frac{2}{\lambda^2}(x-\mu)e^{-(\frac{x-\mu}{\lambda})^2} \qquad \mu \le x \le \infty$

Obtain the MLE for μ **and** λ.

Solution:

For μ **:**

Now in the given density limit parameter μ is involved in the lower limit. So $\hat{\mu} = x_{(1)}$ is its MLE.

For λ :

$$f(x) = \frac{2}{\lambda^2}(x-\mu)e^{-(\frac{x-\mu}{\lambda})^2}$$

Taking likelihood function

$$L(\underline{x}) = \frac{2^n}{\lambda^{2n}}\prod_{i-1}^{n}(x-\mu)e^{-\Sigma(\frac{x-\mu}{\lambda})^2}$$

Taking log likelihood function

$$\log L(\underline{x}) = n\log 2 - 2n\log\lambda + \sum\log(x-\mu) - \frac{\sum(x-\mu)^2}{\lambda^2}$$

Differentiate with respect to 'λ' and put equal to zero.

$$\frac{\partial \log L(\underline{x})}{\partial\lambda} = -\frac{2n}{\lambda} + \frac{2\sum(x-\mu)^2}{\lambda^3}$$

$$0 = \frac{2\sum(x-\mu)^2}{\lambda^3} - \frac{2n}{\lambda}$$

$$\frac{2n}{\lambda} = \frac{2\sum(x-\mu)^2}{\lambda^3}$$

$$\frac{\lambda^3}{\lambda} = \frac{\sum(x-\mu)^2}{\mathrm{n}}$$

$$\lambda^2 = \frac{\sum(x-\mu)^2}{\mathrm{n}}$$

$$\hat{\lambda} = \sqrt{\frac{\sum(x-\mu)^2}{\mathrm{n}}}$$ **This is required result.**

Question # 12

A random sample of size ‘n’, on independent observation is drawn from the

Following density

$$f(x:\alpha,\beta)=\frac{1}{\beta-\alpha}\quad \alpha<x<\beta$$

Find MLE for α & β

Given p.d.f

$$f(x:\alpha,\beta)=\frac{1}{\beta-\alpha}\quad \alpha<x<\beta$$

Since the given parameter involve in the range of density therefore we cannot find

MLE for α & β by using real procedure. In this regard to find MLE for α & β .

i.e

$$\alpha<x_{(1)}<x_{(2)}<x_{(3)}\ldots\ldots\ldots\ldots\ldots<x_{(n)}<\beta$$

So the given MLE or estimators for α & β

$$x_{(1)}=\hat{\alpha}\quad x_{(n)}=\hat{\beta}$$

So the first order statistic is MLE for α and n^{th} order statistic for β

Question #13

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size ‘n’ drawn from a
Bernoulli distribution with p.d.f

$$f(x;p)=\binom{1}{c}_{x} p^{x}(1-p)^{1-x}$$

x: o ,1

Take likelihood function

$$L(x;p)=\underset{i=1}{\overset{n}{\pi}}\binom{1}{c}_{x} p^{\Sigma x}(1-p)^{n-\Sigma x}$$

Take log likelihood function

$$\log L(x;p)=\sum\log\binom{1}{c_x}+\sum x\log p+\left(n-\sum x\right)\log(1-p)$$

Differentiate w.r.t 'p'

$$\frac{\partial\log L(x;p)}{\partial p}=0+\frac{\sum x}{p}+\frac{n-\sum x}{1-p}(-1)$$

$$=\frac{\sum x}{p}-\frac{n-\sum x}{1-p}$$

$$0=\frac{\sum x(1-p)-p\left(n-\sum x\right)}{p(1-p)}$$

$$=\sum x-p\sum x-np+p\sum x$$

$$np=\sum x$$

$$p=\frac{\sum x}{n}$$

$$\hat{p}=\bar{x}$$

Question # 14

Given the C.d.f of logistic distribution

$$F(x)=\left[1+e^{-(\alpha+\beta x)}\right]^{-1}\quad -\infty<x<\infty$$

Obtain MLE for α

As

$$F(x)=\left[1+e^{-(\alpha+\beta x)}\right]^{-1}$$

Differentiate w.r.t 'x'

$$\frac{\partial}{\partial x}F(x)=\frac{d}{dx}\left[1+e^{-(\alpha+\beta x)}\right]^{-1}$$

$$= -\left[1 + e^{-(\alpha+\beta x)}\right]^{-2}\left(0 + e^{-(\alpha+\beta x)}\right)(-\beta)$$

$$= \beta e^{-(\alpha+\beta)}\left[1 + e^{-(\alpha+\beta x)}\right]^{-2}$$

Now its likelihood function is

Now taking log on b.s

$$\log L(x;\beta) = n\log\beta - \sum(\alpha+\beta x)\log e + \sum\log\left[1 + e^{-(\alpha+\beta x)}\right]^{-2}$$

Now differentiate w.r.t, α

$$\frac{d\log L(\underline{x})}{d\alpha} = 0 - n - 2\sum\frac{1}{\left[1 + e^{-(\alpha+\beta x)}\right]}e^{-(\alpha+\beta x)}(-1)$$

$$n = 2\sum\frac{e^{-\alpha-\beta x}}{\left[1 + e^{-(\alpha+\beta x)}\right]}$$

$$n = 2\sum\frac{e^{-\alpha}e^{-\beta x}}{1 + e^{-(\alpha+\beta x)}}$$

$$\frac{n}{2}\sum\frac{1 + e^{-(\alpha+\beta x)}}{e^{-\beta x}} = e^{-\alpha}$$

Taking log on b.s

$$\hat{\alpha} = -\ln\left[\frac{n}{2}\sum\frac{1 + e^{-(\alpha+\beta x)}}{e^{-\beta x}}\right]$$

We differentiate w.r.t ‘ β

$$\frac{\partial\ln\beta}{\partial\beta} = \frac{n}{\beta} - 0 - \sum x - 2\sum\frac{1}{1 + e^{-(\alpha+\beta)}}e^{-(\alpha+\beta x)}(-x)$$

$$= \frac{n}{\beta} - \sum x + 2\sum\frac{xe^{-(\alpha+\beta x)}}{1 + e^{-(\alpha+\beta)}}$$

$$\frac{1}{\beta} = \frac{\sum x}{n} - 2\sum \frac{xe^{-(\alpha+\beta x)}}{\left(1+e^{-(\alpha+\beta x)}\right)}$$

$$\hat{\beta} == \frac{1}{\bar{x} - \frac{2}{n}\sum \frac{xe^{-(\hat{\alpha}+\hat{\beta}x)}}{1+e^{-(\hat{\alpha}+\hat{\beta}x)}}}$$

Question no 15:

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from the distribution *with p.d.f*

$f(X;\theta) = \theta^2 (X+1)(1-\theta)^X \quad \theta : 0,1$

Calculate θ^n the MLE of θ. Use MLE to construct 90% C.I for θ.

Where $E(X) = \frac{2(1-\theta)}{\theta}$

SOLUTION:

Given p.d.f is

$f(X;\theta) = \theta^2 (X+1)(1-\theta)^X$

Take likelihood function

$$L(X;\theta) = \theta^{2n} \underset{i=1}{\overset{n}{\pi}} (X+1)(1-\theta)^{\Sigma X}$$

Take log likelihood function

$$\log L(X;\theta) = \log\left[\theta^{2n} \underset{i=1}{\overset{n}{\pi}} (X+1)(1-\theta)^{\Sigma X}\right]$$

$$\log L(X;\theta) = 2n\log\theta + \sum \log(X+1) + \sum X \log(1-\theta)$$

Differentiate with respect to ‘ θ ’

$$\frac{\partial \log L(X;\theta)}{\partial\theta} = \frac{2n}{\theta} + \frac{\sum X}{(1-\theta)}(-1)$$

$$= \frac{2n}{\theta} - \frac{\sum X}{(1-\theta)} \rightarrow (A)$$

$$0 = \frac{2n(1-\theta) - \theta \sum X}{\theta(1-\theta)}$$

$$0 = 2n - 2n\theta - \theta \sum X$$

$$0 = 2n - 2n\theta - \theta n \bar{X}$$

$$0 = -n(-2 + 2\theta + \theta \bar{X})$$

$$2 = 2\theta + \theta \bar{X}$$

$$2 = \theta(2 + \bar{X})$$

$$\hat{\theta} = \frac{2}{2 + \bar{X}}$$

Again differentiate with respect to 'θ' to equation (A)

$$\frac{\partial^2 \log(X;\theta)}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{\sum X}{(1-\theta)^2}(-1)$$

$$= -\frac{2n}{\theta^2} - \frac{\sum X}{(1-\theta)^2}$$

Applying expectation on both side

$$E\left[\frac{\partial^2 \log L(X;\theta)}{\partial \theta^2}\right] = -\frac{2n}{\theta^2} - \frac{nE(X)}{(1-\theta)^2}$$

$$= -\frac{2n}{\theta^2} - \frac{n(2(1-\theta)/\theta)}{(1-\theta)^2}$$

$$= -\frac{2n}{\theta^2} - \frac{2n}{\theta(1-\theta)}$$

$$= \frac{-2n(1-\theta) - 2n\theta}{\theta^2(1-\theta)}$$

$$= \frac{-2n + 2n\theta - 2n\theta}{\theta^2(1-\theta)}$$

$$E\left[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}\right]=-\frac{2n}{\theta^2(1-\theta)}$$

Now

$$\mathrm{var}(\hat{\theta})=-\frac{1}{-E\left[\frac{\partial^2 \log L(X;\theta)}{\partial\theta^2}\right]}$$

$$=\frac{1}{\frac{2n}{\theta^2(1-\theta)}}$$

$$\mathrm{var}(\hat{\theta})=\frac{\theta^2(1-\theta)}{2n}$$

The 100 (1-α)% C.I for θ

$$\hat{\theta}\pm Z_{\alpha/2}S.E(\hat{\theta})$$

The 90% C.I for θ

$$\hat{\theta}\pm 1.645\sqrt{\frac{(\hat{\theta})^2(1-\hat{\theta})}{2n}}$$

$$\frac{2}{2+\overline{X}}\pm 1.645\sqrt{\frac{(2/2+\overline{x})^2(1-2/2+\overline{x})}{2n}}$$

Which is required result**.**

Question No 16:

Let ‘X’ be a random sample with p.d.f

$$f(X;\mu)=e^{-(X-\mu)-e^{-(X;\mu)}}$$

Suppose that random sample $X_1, X_2, X_3,..., X_n$ is available from the distribution show that MLE for μ is

$\hat{\mu} = -\ln(n^{-1}e^{-\sum X})$

If n is large use the generally large proportion of the MLE to construct an approximate 100%confidence interval for μ Evaluate the interval if $\alpha = 0.10$,n=100 , $\hat{\mu} = 0.05$

Let the given p.d.f

$$f(X;\mu) = e^{-(X-\mu)-e^{-(X;\mu)}}$$

$$= e^{-(X-\mu)}e^{-e^{-(X;\varpi)}}$$

Take likelihood function

$$L(X;\mu) = \underset{i=1}{\overset{n}{\pi}}\left[e^{-(X-\mu)}e^{-e^{-(X-\mu)}}\right]$$

$$= \left[e^{-\sum(X-\mu)}e^{-\sum e^{-(X-\mu)}}\right]$$

Take log likelihood function

$$\log L(X;\mu) = -\sum(X-\mu)\log e - \sum e^{-(X-\varpi)}\log e$$

$$= -\sum X + n\mu - \sum e^{-(X-\mu)}$$

Differentiate with respect to μ

$$\frac{\partial \log L(X;\mu)}{\partial \mu} = n - \sum e^{-(X-\mu)}(1)$$

$$= n - \sum e^{-X}e^{\mu} \rightarrow (A)$$

Equating to zero

$$0 = n - \sum e^{-X}e^{\mu}$$

$$n = \sum e^{-X}e^{\mu}$$

$$e^{\mu} = \frac{n}{\sum e^{-X}}$$

$$\frac{1}{e^{\mu}} = \frac{\sum e^{-X}}{n}$$

$$e^{-\mu} = n^{-1}(\sum e^{-X})$$

Taking log on both sides

$$-\mu = \ln\left[n^{-1}(\sum e^{-X})\right]$$

$$\hat{\mu} = -\ln\left[n^{-1}(\sum e^{-X})\right]$$

Which is required MLE for ‘ μ ’

Again differentiate with respect to ‘ μ ’ to (A)

$$\frac{d^2 \ln L(X;\mu)}{d\mu^2} = -\sum e^{-X} e^{\mu} \therefore e^{\mu} = \frac{n}{\sum e^{-X}} \; \frac{d^2 \ln L(X;\mu)}{d\mu^2} = -\sum e^{-X} \frac{n}{\sum e^{-X}}$$

$$= -n$$

Applying expectation

$$E\left[\frac{\partial^2 \ln L(X;\mu)}{\partial \mu^2}\right] = -n$$

Now

$$\operatorname{var}(\hat{\mu}) = -\frac{1}{-E\left[\frac{\partial^2 \log L(X;\mu)}{\partial \mu^2}\right]}$$

$$\operatorname{var}(\hat{\mu}) = \frac{1}{n}$$

The 100(1- α)% C.I for μ

$$\hat{\mu} \pm Z_{\alpha/2} S.E(\hat{\mu})$$

The 90% C.I for’ μ ’ is

$$0.05 \pm 1.645 \frac{1}{\sqrt{100}}$$

$$0.05 \pm 1.645 \frac{1}{10}$$

$$(-0.1145, 0.2145)$$

We are stating that 90% C.I starts from (-0.1145) to (0.2145) shall contain the true value of population parameter μ

Question No 17:

Given that

$$f(X;\mu) = \alpha e^{-\alpha(X-\mu)-e^{-\alpha(X-\mu)}}$$

Obtain the MLE for α and μ

Where

$$e^{-\hat{\alpha}\hat{\mu}} = \frac{1}{n}\sum e^{-\hat{\alpha}Xi}$$

The given p.d.f is

$$f(X;\mu) = \alpha e^{-\alpha(X-\mu)-e^{-\alpha(X-\mu)}}$$

Take likelihood function

$$L(X;\mu) = \alpha^{n} e^{-\alpha\sum(X-\mu)e^{-\sum e^{-\alpha(x-\mu)}}}$$

Taking log likelihood function

$$\log L(X;\mu) = n\log\alpha - \alpha\sum(X-\mu) - \sum e^{-\alpha(X-\mu)}$$

$$= n\log\alpha - \alpha n\overline{X} + \alpha n\mu - \sum e^{-\alpha(X-\mu)} \therefore -(X-\mu)$$

$$= -X + \mu$$

$$= \mu - X$$

Differentiate with respect to α

$$\frac{\partial \log L(X;\mu)}{\partial \alpha} = \frac{n}{\alpha} - n\overline{X} + n\mu - \sum e^{-\alpha(X-\mu)}(\mu - X)$$

$$= \frac{n}{\alpha} - n\overline{X} + n\mu - \sum e^{-\alpha X} e^{\alpha\mu}(\mu - X)$$

$$= \frac{n}{\hat{\alpha}} - n\overline{X} + n\mu - \hat{\mu} e^{\alpha\mu} \sum e^{-\hat{\alpha}X} + e^{\alpha\mu} \sum e^{-\hat{\alpha}x} X$$

$$= \frac{n}{\hat{\alpha}} - n\overline{X} + n\mu - e^{\alpha\mu}(\hat{\mu} \sum e^{-\hat{\alpha}X} + \sum X e^{-\hat{\alpha}x})$$

Hint:

$$e^{-\alpha\mu} = \frac{1}{n} \sum e^{-\alpha Xi}$$

$$\frac{n}{\sum e^{-\alpha Xi}} = \frac{1}{e^{-\alpha\mu}} = e^{\alpha\mu}$$

$$= \frac{n}{\hat{\alpha}} - n\overline{X} + n\mu - \frac{n}{\sum e^{-\alpha Xi}}(\hat{\mu} \sum e^{-\hat{\alpha}X} + \sum X e^{-\hat{\alpha}x})$$

$$\frac{\partial \log L(X;\mu)}{\partial \alpha} = \frac{n}{\hat{\alpha}} - n\overline{X} + n\mu - n(\hat{\mu} - \frac{\sum X e^{-\hat{\alpha}X}}{\sum e^{-\hat{\hat{\alpha}}Xi}})$$

$$0 = \frac{n}{\hat{\alpha}} - n\overline{X} + n\mu - n\hat{\mu} + n\frac{\sum X e^{-\hat{\alpha}Xi}}{\sum e^{-\hat{\alpha}Xi}}$$

$$0 = \frac{n}{\hat{\alpha}} - n\overline{X} + n\frac{\sum X e^{-\hat{\alpha}Xi}}{\sum e^{-\hat{\alpha}Xi}}$$

$$\frac{n}{\hat{\alpha}} = n(\overline{X} - \frac{\sum X e^{-\hat{\alpha}Xi}}{\sum e^{-\hat{\varepsilon}Xi}})$$

$$\frac{1}{\hat{\alpha}} = \overline{X} - \frac{\sum Xe^{-\hat{\alpha}Xi}}{\sum e^{-\hat{\alpha}Xi}})$$

Again differentiate with respect to ‘ μ ’ equation (A)

$$\frac{\partial^2 \log L(X;\mu)}{\partial\mu} = 0 - 0 + n\alpha - \alpha\sum e^{-\alpha(X-\mu)}$$

$$n\alpha = \alpha\sum e^{-\alpha(X-\mu)}$$

$$n = \sum e^{-\alpha X} e^{\alpha\mu}$$

$$e^{\alpha\mu} = \frac{n}{\sum e^{-\alpha X}}$$

$$e^{-\alpha\mu} = \frac{\sum e^{-\alpha X}}{n}$$

$$e^{-\alpha\mu} = n^{-1}(\sum e^{-\alpha X})$$

Applying natural log on b.s

$$-\alpha\mu = \ln\left[n^{-1}(\sum e^{-\alpha X})\right]$$

$$\hat{\mu} = \frac{-\ln\left[n^{-1}\sum e^{-\alpha X}\right]}{\alpha}$$

Which is required MLE for’ μ ’

Question NO 18:

Obtain the MLE for θ^2 and θ from the density of Raleigh distribution. Also show that $\hat{\theta}^2$ is most efficient and sufficient and MVBUE?

Solution:

As we know that

$$f(X)=\frac{2X}{\theta^2}e^{-(X/\theta)^2}$$

Taking likelihood function

$$L(X)=\frac{2^n}{\theta^{2n}}\pi_{i=1}^{n}Xe^{-\Sigma(X/\theta)^2}$$

Taking log likelihood function

$$\log L(X)=\log\left[\frac{2^n}{\theta^{2n}}\pi_{i=1}^{n}Xe^{-\Sigma(X/\theta)^2}\right]$$

$$=\log 2^n-\log\theta^{2n}+\sum\log X-\sum(X/\theta)^2$$

$$=n\log 2-2n\log\theta+\sum\log X-\sum X^2/\theta^2$$

Differentiate with respect to θ

$$\frac{\partial\log L(X)}{\partial\theta}=0-\frac{2n}{\theta}+2\sum\frac{X^2}{\theta^3}$$

$$\frac{2n}{\theta}=\frac{2\sum X^2}{\theta^3}$$

$$n=\frac{\sum X^2}{\theta^2}$$

$$n\theta^2=\sum X^2$$

$$\theta^2=\frac{\sum X^2}{n}$$

MLE for θ^2

$$\hat{\theta}^2=\frac{\sum X^2}{n}$$

MLE for θ

$$\hat{\theta} = \sqrt{\frac{\sum X^2}{n}}$$

For unbiasedness

$$E(\hat{\theta}^2) = \frac{\sum E(X^2)}{n} \therefore E(X^2) = \theta^2$$

$$= \frac{n\theta^2}{n}$$

$$E(\hat{\theta}^2) = \theta^2$$

And

$$E(\hat{\theta}) = E\left[\sqrt{\frac{\sum X^2}{n}}\right]$$

$$E(\hat{\theta}) \neq \theta$$

$\hat{\theta}^2$ is an unbiased estimator of θ^2. As invariance and unbiased property cannot be satisfied on θ. In general $\hat{\theta}$ is biased estimator for θ.

For MVBUE:

$$f(X) = \frac{2X}{\theta^2} e^{-(x/\theta)^2}$$

Taking likelihood function

$$L(X) = \frac{2^n}{\theta^{2n}} \pi_{i=1}^{n} X e^{-\sum(x/\theta)^2}$$

Taking log likelihood function

$$\log L(X) = \log\left[\frac{2^n}{\theta^{2n}} \pi_{i=1}^{n} X e^{-\sum(x/\theta)^2}\right]$$

$$= \log 2^n - \log \theta^{2n} + \sum \log X - \sum(x/\theta)^2$$

$$= n\log 2 - 2n\log\theta + \sum \log X - \sum X^2/\theta^2$$

Differentiate with respect to θ

$$\frac{\partial \log L(X)}{\partial\theta} = 0 - \frac{2n}{\theta} + 0 + \sum \frac{X^2}{\theta^3}$$

$$= -\frac{2n}{\theta} + \frac{2\sum X^2}{\theta^3}$$

$$= \frac{2n}{\theta^3}\left[\frac{\sum X^2}{n} - \theta^2\right] \rightarrow (1)$$

As we know that

$$\frac{\partial \log L(X)}{\partial\theta} = A(\theta)\left[\hat{\theta} - \tau(\theta)\right] \rightarrow (2)$$

Comparing equation (1) and (2)

$$A(\theta) = \frac{2n}{\theta^3}$$

$$\hat{\theta} = \frac{\sum X^2}{n} \qquad , \tau(\theta) = \theta^2 \qquad , \tau'(\theta) = 2\theta$$

Now

$$Var(\hat{\theta}) = \frac{\tau'(\theta)}{A(\theta)}$$

$$= \frac{2\theta}{\frac{2n}{\theta^3}}$$

$$= \frac{\theta}{\frac{n}{\theta^3}}$$

$$Var(\hat{\theta}) = \frac{\theta^4}{n}$$

$\hat{\theta}^2$ is sufficient statistic and is the MVBUE.

Theorem:

Prove that asymptotic property of MLE?

OR

MLE tends to normality as $n \to \infty$

OR

Show that for $n \to \infty$ than MLE follows normal distribution.

Let we have a likelihood function $L(X;\theta)$ and assuming that its derivative exist.

We know that

$$\int \cdots \int L(X;\theta) d\underline{X} = 1$$

Partially differentiate with respect to 'θ'

$$\frac{\partial}{\partial \theta}\left[\int \cdots \int L(X;\theta) d\underline{X}\right] = \frac{\partial}{\partial \theta}(1)$$

$$\int \cdots \frac{\partial}{\partial \theta} \int L(X;\theta) d\underline{X} = 0$$

$$\int \cdots \int \frac{\partial}{\partial \theta} L(X;\theta) d\underline{X} = 0$$

$$\int \cdots \int \frac{\partial}{\partial \theta} L(X;\theta) \frac{L(X;\theta)}{L(X;\theta)} d\underline{X} = 0$$

$$\int \cdots \int \frac{1}{L(X;\theta)} \frac{\partial}{\partial \theta} L(X;\theta) L(X;\theta) d\underline{X} = 0$$

$$\int \cdots \int \frac{\partial}{\partial \theta} \ln L(X;\theta) L(X;\theta) d\underline{X} = 0$$

$$E\left[\frac{\partial}{\partial \theta} \ln L(X;\theta)\right] = 0 \to (A)$$

Now using Taylor Theorem we get

$$\left[\frac{\partial}{\partial \theta} \ln L(X;\theta)\right]_{\hat{\theta}} = \left[\frac{\partial}{\partial \theta} \ln L(X;\theta)\right]_{\theta_0} + (\hat{\theta} - \theta_0)\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]_{\theta^*} \to (B)$$

Where θ^* is the same value between $\hat{\theta} \& \theta_0$

As $\hat{\theta}$(MLE) is a solution of $\frac{\partial \ln L(X;\theta)}{\partial\theta} = 0$

Then equation B will be

$$0 = \left[\frac{\partial}{\partial\theta}\ln L(X;\theta)\right]_{\theta_0} + (\hat{\theta}-\theta_0)\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right]_{\theta^*} \rightarrow (C)$$

As $n \to \infty$ each converges to its expectation by the law of large number

i.e $$n \rightarrow \infty(\frac{\partial \ln L}{\partial\theta}) = E\left[\frac{\partial \ln L}{\partial\theta}\right] = 0$$

And

$$n \rightarrow \infty(\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}) = E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right] \neq 0$$

$\hat{\theta}-\theta_0$ converges to zero as $n \to \infty$ it means that $\hat{\theta}$ is consistant estimator of θ.

Now consider equation (C)

$$-\left[\frac{\partial \ln L}{\partial\theta}\right]_{\theta_0} = (\hat{\theta}-\theta_0)E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right]_{\theta^*}$$

$$(\hat{\theta}-\theta_0) = \frac{\left[\frac{\partial \ln L(X;\theta)}{\partial\theta}\right]_{\theta_0}}{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right]\theta^*}$$

$$(\hat{\theta}-\theta_0) = \frac{\left[\frac{\partial \ln L(X;\theta)}{\partial\theta}\right]_{\theta_0}}{\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right]}\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial\theta^2}\right]}}$$

$$\frac{\left[\frac{\partial \ln L(X;\theta)}{\partial \theta}\right]_{\theta_0}}{\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]}} = \frac{\hat{\theta}-\theta_0}{\frac{1}{\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]}}} \rightarrow (D)$$

By central limit Theorem

$$\frac{\left[\frac{\partial \ln L(X;\theta)}{\partial \theta}\right]_{\theta_0}}{\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]}} \approx S.N.V$$

So equation (D)

$$\frac{\hat{\theta}-\theta_0}{\frac{1}{\sqrt{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]\theta^*}}} \approx S.N.V$$

i.e

$$\hat{\theta} \approx N\left(\theta_0, \frac{1}{-E\left[\frac{\partial^2 \ln L(X;\theta)}{\partial \theta^2}\right]}\theta^*\right)$$

Required result